

Design of an Experiment to Test Nonclassical Probabilistic Behavior of the Financial market

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Abstract

The recent crash demonstrated (once again) that the description of the financial market by present financial mathematics cannot be considered as totally satisfactory. We remind that nowadays financial mathematics is heavily based on the use of random variables and stochastic processes which are described by Kolmogorov's measure-theoretic model for probability ("classical probabilistic model"). I speculate that the present financial crises is a sign (a kind of experiment to test validity of classical probability theory at the financial market) that the use of this model in finances should be either totally rejected or at least completed. One of the best candidates for a new probabilistic financial model is quantum probability or its generalizations, so to say quantum-like (QL) models. Speculations that the financial market may be nonclassical have been present in scientific literature for many years. The aim of this note is to move from the domain of speculation to rigorous statistical arguments in favor of probabilistic nonclassicality of the financial market. I design a corresponding statistical test which is based on violation of the formula of total probability (FTP). The latter is the basic in classical probability and its violation would be a strong sign in favor of QL behavior at the market.

1 Introduction

Detailed review on applications of quantum mathematics to finances can be found in author's monograph [1], we mention just some publications [1]–[8].

The aim of this paper was formulated in the abstract. We point out that the experimental test to check a possibility of violation of FTP at the financial market can be considered as adaptation to finances of the general statistical test proposed in [9]. Its version was already tested in cognitive science, see Conte et al. [10], [11]. It was shown that FTP (and hence classical probability theory) is violated in some experiments on recognition of ambiguous pictures.

Our experiment may be criticized by dealers working at the real market. We cannot exclude such a possibility. However, our experiment opens the door toward designing of similar may be more realistic financial experiments. As a first step, one may try to perform our experiment with students.

2 Supplementary (“Complementary”) Stocks

In ordinary QM one pays a lot of attention to so called *complementary observables*; for example, position and momentum. Since we will operate with discrete observables, we can mention electron's spin projections to two different directions or photon's polarization projections as examples of complementary observables. Complementary observables are represented in QM formalism by noncommuting operators.

In QL-finances, we will also operate with observables which are represented by noncommuting operators.

The delicate point is that, unlike QM, at the moment we do not have a quantization procedure for financial variables – to produce from variables operators. In a series of papers, e.g., [12], books [1], [15], I developed a kind of quantization procedure: starting with probabilities one can produce operator representation of variables. Therefore it would be useful to formulate in pure probabilistic terms condition of noncommutativity.

Consider two observables a and b in QM. Suppose that they are dichotomous, $a \in X_a = \{\alpha_1, \alpha_2\}$ and $b \in X_b = \{\beta_1, \beta_2\}$. They can be represented in QM formalism by self-adjoint operators in the two

dimensional complex space, $\mathcal{H}_2 = \mathbf{C} \times \mathbf{C}$; thus by 2×2 Hermitian matrices: \hat{a}, \hat{b} . We recall that any symmetric matrix can be diagonalized in the basis consisting of its eigenvectors, say

$$\hat{a}e_\alpha^a = \alpha e_\alpha^a, \quad \alpha = \alpha_1, \alpha_2;$$

$$\hat{b}e_\beta^b = \beta e_\beta^b, \quad \beta = \beta_1, \beta_2.$$

The crucial point is that if operators do not commute, then they cannot be diagonalized in the same basis. It means that

$$\langle e_\alpha^a, e_\beta^b \rangle \neq 0,$$

for any pair of values α, β . Conditional (transition) probabilities can be expressed via scalar products of corresponding eigenvectors:

$$\mathbf{P}(b = \beta | a = \alpha) = |\langle e_\alpha^a, e_\beta^b \rangle|^2.$$

Thus two observables are complementary iff all these probabilities are strictly positive:

$$\mathbf{P}(b = \beta | a = \alpha) > 0 \tag{1}$$

for all α and β . The latter condition has no direct relation to QM. It can be used in any domain of science.

We now analyze little bit the meaning of this condition. It can be equivalently written as

$$\mathbf{P}(b = \beta | a = \alpha) \neq 1 \tag{2}$$

Thus it is impossible to determine a value $b = \beta$ by fixing the value $a = \alpha$. The b -variable has some features which cannot be determined on the basis of features of the a -variable. Thus b contains additional, or to say supplementary, information. Therefore we call observables (from any domain of science) *supplementary* if (1) holds. We may call them complementary as Bohr did in QM. But, unlike Bohr, we do not emphasize *mutual exclusivity of measurements*. In principle, supplementary observables, unlike complementary, can be measured simultaneously. However, supplementary observables are also mathematically represented by noncommuting operators. We recall that our aim is to show the adequacy of the mathematical apparatus of QM to the financial market. Thus, we need not borrow even quantum ideology and philosophy.

We will consider supplementary stocks. Formally, one can determine either two stocks, say A and B , are supplementary or not by using the formal definition (1). However, to do this, we should perform experiment for a large ensemble of dealers. If, finally, one observes that transition frequencies are close to zero, it will imply that this pair of stocks is not useful for coming interference experiment. Therefore it is much better to use financial intuition to determine either two stocks are intended to be supplementary or not.

3 Experiment Design

- 1). Select two supplementary stocks, say A and B .
- 2). Select of an ensemble Ω of dealers who used to work with these two stocks. Its size N should be large enough
- 3). Select an interval δ giving average time between two successive financial operations.¹
- 4). Define two observables: for a dealer $\omega \in \Omega$, $a(\omega) = +1$ if he has bought a packet of stocks during the period δ and $a(\omega) = -1$ if he has not.² The b -observable is defined in the same way.
- 4). Starting with some initial instant of time, say t_0 , wait until $t_0 + \delta$. Then count all dealers who has bought during this period some packet³ of A -stocks, i.e., all elements $\omega \in \Omega$ such that $a = +1$. Denote this number by $n^a(+)$. We define frequency probabilities

$$p^a(+) = n^a(+)/N, p^a(-) = 1 - p^a(+);$$

In the same way we find $n^b(+)$ – the number of dealers whose B -bids has matched asks at the market (during the same period $[t_0, t_0 + \delta]$ – and define frequency probabilities $p^b(\pm)$.

- 5). On the basis of previous a -measurement select from Ω sub-ensembles of dealers Ω_+^a – those whose A -bids were realized during the period $[t_0, t_0 + \delta]$ – and Ω_-^a – those whose A -bids did not match

¹If during some period of time T (e.g., depending of frequency of operating, one day, or month, or year), a dealer made k operations at the financial market, then δ is equal to average of T/k with respect to all dealers from the ensemble Ω selected for the experiment.

²Even if his A -bid was present at the market, but it did not match asked prices for this stock; $a = -1$ as well if he did not submit any A -bid.

³In the experiment under consideration the size of packet does not play any role. However, experiment can be design in a more complicated way, by including the size of a packet. In this way nonsignificant bids can be excluded from the game.

any asked price for the A -stock or those who did not bid anything for this stock. Denote the numbers of elements in these ensembles by N_+^a and N_-^a , respectively.

6). Wait until $t_0 + 2\delta$ and after this count all dealers from Ω_+^a whose B -bids were realized during the period $[t_0 + \delta, t_0 + 2\delta]$. These are elements $\omega \in \Omega_+^a$ for whom $b(\omega) = +1$. Denote obtained number by $n(+|+)$. We define frequency probabilities

$$p(+|+) = n(+|+)/N_+^a, \quad p(-|+) = 1 - p(+|+).$$

They have the meaning of conditional probabilities. For example, $p(+|+)$ is probability that a randomly chosen dealer first bought a packet of the A -stocks and then a packet of the B -stocks.

In same way we define frequency probabilities

$$p(+|-) = n(+|-)/N_-^a, \quad p(-|-) = 1 - p(+|-)$$

by making the b -measurement for dealers belonging to the sub-ensemble Ω_-^a .

6). Finally, define the coefficient

$$\lambda_\beta = \frac{p^b(\beta) - [p^a(+)p(\beta|+) - p^a(-)p(\beta|-)]}{2\sqrt{p^a(+)p(\beta|+)p^a(-)p(\beta|-)}}, \quad \text{where } \beta = \pm. \quad (3)$$

It gives a measure of deviation from the classical formula of total probability, see Section 4. In quantum mechanics this coefficient has the form

$$\lambda_\beta = \cos \theta_\beta$$

where θ_β is the phase angle. Therefore it can be called *interference coefficient*.

7). An empirical situation with $\lambda \neq 0$ would yield evidence for QL behavior of the financial market: interaction of dealers and stocks. In this case, starting with the (experimentally calculated) coefficient of interference λ we can proceed either to the conventional Hilbert space formalism (if this coefficient is bounded by 1) or to so called hyperbolic Hilbert space formalism (if this coefficient is larger than 1), see [9], [12] and more in coming book [15].

4 Formula of Total Probability with Interference Term

In the above notations the conventional (“classical”) formula of total probability (FTP) is written as

$$p^b(\beta) = p^a(+)p(\beta|+) - p^a(-)p(\beta|-). \quad (4)$$

Thus the probability $p^b(\beta)$ can be found on the basis of conditional probabilities $p(\beta|\pm)$.⁴ FTP plays the fundamental role in modern science. Its consequences are strongly incorporated in modern scientific reasoning. It is derived in classical probability theory by using Kolmogorov measure-theoretic model for probability. This model is the basis of modern financial mathematics which is based on the use of classical random variables (and stochastic processes).

In [1] I pointed out that the quantum formalism induces a violation of FTP. An additional term appears in the right hand side of (4), so called *interference term*. Violation of the law of total probability can be considered as an evidence that the classical probabilistic description could not be applied. The λ -coefficient (3) gives us a measure of statistical deviation from FTP.

Our aim is to show that QL probabilistic descriptions could be applied. The terminology “quantum-like” and not simply “quantum” is used to emphasize that violations of (4) are not reduced to those which can be described by the conventional quantum model. In particular, statistical data from the financial market may be described by a generalized quantum formalism, see for details [1], [12] and especially [15].

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⁴ “The prior probability to obtain the result e.g. $b = +$ is equal to the prior expected value of the posterior probability of $b = +$ under conditions $a = \pm$.”

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